On geometric SDPS-sets of elliptic dual polar spaces

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Abstract

Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$. Let $Q^-(2n+1, \mathbb{K})$ be a nonsingular quadric of Witt index n in $\mathrm{PG}(2n+1, \mathbb{K})$ whose associated quadratic form defines a nonsingular quadric $Q^+(2n+1, \mathbb{K}')$ of Witt index $n+1$ in $\mathrm{PG}(2n+1, \mathbb{K}')$. For even n, we define a class of SDPS-sets of the dual polar space $DQ^-(2n+1, \mathbb{K})$ associated to $Q^-(2n+1, \mathbb{K})$, and call its members geometric SDPS-sets. We show that geometric SDPS-sets of $DQ^-(2n+1, \mathbb{K})$ are unique up to isomorphism and that they all arise from the spin embedding of $DQ^-(2n+1, \mathbb{K})$. We will use geometric SDPS-sets to describe the structure of the natural embedding of $DQ^-(2n+1, \mathbb{K})$ into one of the half-spin geometries for $Q^+(2n+1, \mathbb{K}')$.

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1 Introduction

Let $n \in \mathbb{N} \setminus \{0, 1\}$, let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$ and let θ denote the unique nontrivial element in $Gal(\mathbb{K}'/\mathbb{K})$. Let $Q^-(2n+1, \mathbb{K})$ be a nonsingular quadric of Witt index n in $\mathrm{PG}(2n+1, \mathbb{K})$ whose associated quadratic form defines a nonsingular quadric $Q^+(2n+1, \mathbb{K}')$ of Witt index $n+1$ in $\mathrm{PG}(2n+1, \mathbb{K}')$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two systems of generators (= maximal subspaces) of $Q^+(2n+1, \mathbb{K}')$. Recall that

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two generators belong to the same system if they intersect in a subspace of even co-dimension. For every $\epsilon \in \{+,-\}$, let $HS^\epsilon(2n+1,\mathbb{K}')$ denote the point-line geometry whose points are the elements of $\mathcal{M}^\epsilon$ and whose lines are the $(n-2)$-dimensional subspaces of $Q^+(2n+1,\mathbb{K}')$ (natural incidence). The isomorphic geometries $HS^+(2n+1,\mathbb{K}')$ and $HS^-(2n+1,\mathbb{K}')$ are called the *half-spin geometries* for $Q^+(2n+1,\mathbb{K}')$. Let $DQ^-(2n+1,\mathbb{K})$ denote the dual polar space associated to the quadric $Q^-(2n+1,\mathbb{K})$. The map which associates with every generator of $Q^-(2n+1,\mathbb{K})$ the unique element of $\mathcal{M}^\epsilon$ containing it, defines a full embedding of $DQ^-(2n+1,\mathbb{K})$ into $HS^\epsilon(2n+1,\mathbb{K}')$, see Cooperstein and Shult [6] (for the finite case) and De Bruyn [9] (general case). This full embedding is called the *natural embedding* of $DQ^-(2n+1,\mathbb{K})$ into $HS^\epsilon(2n+1,\mathbb{K}')$.

An SDPS-set of a dual polar space Δ of rank $2n'$ is a very nice set of points of Δ carrying the structure of a dual polar space of rank n' (see Section 2). SDPS-sets of dual polar spaces were introduced by De Bruyn and Vandecasteele [11] because of their connection with the theory of valuations of near polygons. From that connection, it follows that the set of points of Δ at non-maximal distance from a given SDPS-set X is a hyperplane of Δ. We call this hyperplane the *hyperplane of Δ associated to X*.

In Section 4, we will construct a certain class of SDPS-sets of $DQ^-(2n+1,\mathbb{K})$, n even. The construction is as follows. Let α be a generator of $Q^+(2n+1,\mathbb{K}')$ which is disjoint from its conjugate α^θ (with respect to the quadratic extension $\mathbb{K}'$ of $\mathbb{K}$). Let H denote the following set of points of α: a point x of α belongs to H if and only if x is collinear on $Q^+(2n+1,\mathbb{K}')$ with its conjugate x^θ. Then H is a nonsingular Hermitian variety of Witt index $\frac{n}{2}$ of α.

Theorem 1.1 *If β is a generator of H, then $\langle\beta,\beta^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$ is a generator of $Q^-(2n+1,\mathbb{K})$. The set of generators of $Q^-(2n+1,\mathbb{K})$ which can be obtained in this way is an SDPS-set of $DQ^-(2n+1,\mathbb{K})$.*

Any SDPS-set of $DQ^-(2n+1,\mathbb{K})$, n even, which can be obtained as described in Theorem 1.1 is called *geometric*. We prove the following in Section 4.

Theorem 1.2 *Up to isomorphism, there exists a unique geometric SDPS-set in $DQ^-(2n+1,\mathbb{K})$, n even and $n \geq 2$.*

The following theorem provides information regarding the structure of the natural embedding of $DQ^-(2n+1,\mathbb{K})$ into one of the half-spin geometries for $Q^+(2n+1,\mathbb{K}')$. We will prove it in Section 5.

Theorem 1.3 *Consider the natural embedding of* $\Delta = DQ^-(2n+1,\mathbb{K})$ *into* $HS^\epsilon(2n+1,\mathbb{K}')$, $\epsilon \in \{+,-\}$. *Let* $d_\epsilon(\cdot,\cdot)$ *and* $d_\Delta(\cdot,\cdot)$ *denote the distance functions in the respective geometries* $HS^\epsilon(2n+1,\mathbb{K}')$ *and* Δ. *Then for every point* x *of* $HS^\epsilon(2n+1,\mathbb{K}')$, *there exists a* $K \in \mathbb{N}$ *and a geometric SDPS-set* X *in a convex subspace of diameter* $2K$ *of* $DQ^-(2n+1,\mathbb{K})$ *such that* $d_\epsilon(x,y) = \lfloor \frac{K+1+d_\Delta(X,y)}{2} \rfloor$ *for every point* y *of* Δ.

By [6] and [9], the dual polar space $DQ^-(2n+1,\mathbb{K})$ has a nice full embedding e into the projective space $\mathrm{PG}(2^n-1,\mathbb{K}')$, called the *spin embedding* of $DQ^-(2n+1,\mathbb{K})$. If π is a hyperplane of $\mathrm{PG}(2^n-1,\mathbb{K}')$, then the set of all points x of $DQ^-(2n+1,\mathbb{K})$ for which $e(x) \in \pi$ is a hyperplane of $DQ^-(2n+1,\mathbb{K})$. Hyperplanes of $DQ^-(2n+1,\mathbb{K})$ which can be obtained in this way are said to *arise from* e. In Section 5, we will also prove the following result.

Theorem 1.4 *The hyperplanes of* $DQ^-(2n+1,\mathbb{K})$, n *even, associated to geometric SDPS-sets arise from the spin embedding of* $DQ^-(2n+1,\mathbb{K})$.

Remark. An SDPS-set of $DQ^-(5,\mathbb{K})$ is nothing else than an ovoid of the generalized quadrangle $DQ^-(5,\mathbb{K})$. For any field $\mathbb{K}$, there are ovoids in $DQ^-(5,\mathbb{K})$ which do not arise from the spin embedding, see e.g. Payne & Thas [14, p. 57] for the finite case and De Bruyn & Cardinali [4, Theorem 1.7] for the infinite case. So, an SDPS-set of $DQ^-(5,\mathbb{K})$ is not always geometric. It is still an open problem whether every SDPS-set of $DQ^-(4m+1,\mathbb{K})$, $m \geq 2$, is geometric.

2 Preliminaries

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point on L nearest to x. Here, distances are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called *generalized quadrangles*.

If $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ is a near polygon, then the distance between two points x and y of $\mathcal{S}$ will be denoted by $\mathrm{d}(x,y)$. The set of points at distance $i \in \mathbb{N}$ from a given point $x \in \mathcal{P}$ will be denoted by $\Gamma_i(x)$. If $x \in \mathcal{P}$ and $\emptyset \neq X \subseteq \mathcal{P}$, then $\mathrm{d}(x,X) := \min\{\mathrm{d}(x,y) \,|\, y \in X\}$.

A subspace S of a near polygon $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of S is also contained in S. The points and lines contained in a convex subspace of $\mathcal{S}$ define a sub-near-polygon of $\mathcal{S}$. Convex subspaces of diameter d' are therefore also called *convex sub-$2d'$-gons*. A convex subspace F of $\mathcal{S}$ is called *classical* in $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a necessarily unique point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least 2 common neighbours. If x and y are two points of a dense near $2d$-gon at distance $d' \in \{0, \ldots, d\}$ from each other, then by Theorem 4 of Brouwer and Wilbrink [1], x and y are contained in a unique convex subspace $\langle x, y\rangle$ of diameter d'. These convex subspaces are called *quads* if $d' = 2$, *hexes* if $d' = 3$ and *maxes* if $d' = d - 1$.

A function f from the point-set of a dense near $2n$-gon $\mathcal{S}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties:

(V1) $f^{-1}(0) \neq \emptyset$;

(V2) every line L of $\mathcal{S}$ contains a necessarily unique point x_L such that $f(x) = f(x_L) + 1$ for every point $x \in L \setminus \{x_L\}$;

(V3) every point x of $\mathcal{S}$ is contained in a necessarily unique convex subspace F_x such that the following properties are satisfied for every $y \in F_x$: (i) $f(y) \leq f(x)$; (ii) if z is a point collinear with y such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of dense near polygons were introduced in De Bruyn and Vandecasteele [10]. We describe three constructions for obtaining valuations of a given dense near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$.

(1) For every point x of $\mathcal{S}$, the map $f_x : \mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x, y)$ is a valuation of $\mathcal{S}$. We call f_x a *classical valuation* of $\mathcal{S}$.

(2) Suppose O is an *ovoid* of $\mathcal{S}$, i.e. a set of points of $\mathcal{S}$ meeting each line in a unique point. For every point x of $\mathcal{S}$, we define $f_O(x) := 0$ if $x \in O$ and $f_O(x) := 1$ otherwise. Then f_O is a valuation of $\mathcal{S}$. We call f_O an *ovoidal valuation* of $\mathcal{S}$.

(3) Let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a convex sub-near-polygon of $\mathcal{S}$ which is classical in $\mathcal{S}$. Suppose that $f' : \mathcal{P}' \to \mathbb{N}$ is a valuation of F. Then the map $f : \mathcal{P} \to \mathbb{N}; x \mapsto f(x) := \mathrm{d}(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $\mathcal{S}$. We call f the *extension* of f'. If $F = \mathcal{S}$, then the extension is called *trivial*.

Valuations can also induce others.

Proposition 2.1 ([10, Proposition 2.12]) *Let f be a valuation of a dense near polygon $\mathcal{S}$, let F be a convex subspace of $\mathcal{S}$ and let m denote the minimal* *value attained by $f(x)$ as x ranges over all points of F. For every point x of F, we define $f_F(x) := f(x) - m$. Then f_F is a valuation of F.*

The valuation f_F defined in Proposition 2.1 is called the valuation of F *induced* by f.

We now describe an important class of near polygons. Let Π be a nondegenerate polar space (Veldkamp [18]; Tits [17, Chapter 7]) of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The geometry Δ is called a *dual polar space of rank* n and is an example of a near $2n$-gon (Cameron [3]). There exists a bijective correspondence between the nonempty convex subspaces of Δ and the possibly empty singular subspaces of Π. If α is a singular subspace of Π, then the set of all maximal singular subspaces of Π containing α is a convex subspace of Δ. Conversely, every convex subspace of Δ is obtained in this way. Every convex subspace of Δ is classical in Δ. The point-line geometry induced on a convex subspace of diameter $n' \geq 2$ of Δ is a dual polar space of rank n'. If α_1 and α_2 are two maximal singular subspaces of Π, then the distance between α_1 and α_2 in the dual polar space Δ is equal to $n - 1 - \dim(\alpha_1 \cap \alpha_2)$.

In the present paper, we will meet 3 classes of (dual) polar spaces. Let $n \geq 2$, let $\mathbb{K}$ and $\mathbb{K}'$ be two fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$ and let θ be the unique nontrivial element in $Gal(\mathbb{K}'/\mathbb{K})$.

(I) We denote by $Q^-(2n+1, \mathbb{K})$ a nonsingular quadric of Witt index n in $\mathrm{PG}(2n+1, \mathbb{K})$ whose associated quadratic form defines a nonsingular quadric $Q^+(2n+1, \mathbb{K}')$ of Witt index $n+1$ in $\mathrm{PG}(2n+1, \mathbb{K}')$. With respect to a suitable reference system in $\mathrm{PG}(2n+1, \mathbb{K})$, $Q^-(2n+1, \mathbb{K})$ has equation $X_0^2 + (\delta + \delta^\theta) X_0 X_1 + \delta^{\theta+1} X_1^2 + X_2 X_3 + \cdots + X_{2n} X_{2n+1} = 0$, where δ is some element of $\mathbb{K}' \setminus \mathbb{K}$. We denote by $DQ^-(2n+1, \mathbb{K})$ and $DQ^+(2n+1, \mathbb{K}')$ the dual polar spaces associated to $Q^-(2n+1, \mathbb{K})$ and $Q^+(2n+1, \mathbb{K}')$, respectively. We will call $(D)Q^-(2n+1, \mathbb{K})$ an *elliptic (dual) polar space* and $(D)Q^+(2n+1, \mathbb{K}')$ a *hyperbolic (dual) polar space*. (Notice that we have extended this terminology from the finite case to the infinite case.)

(II) We denote by $H(2n, \mathbb{K}', \theta)$ a nonsingular θ-Hermitian variety of Witt index n in $\mathrm{PG}(2n, \mathbb{K}')$ and by $DH(2n, \mathbb{K}', \theta)$ the dual polar space associated to $H(2n, \mathbb{K}', \theta)$. (With *$\theta$-Hermitian* we mean that the associated involutory automorphism is equal to θ.) With respect to a suitable reference system in $\mathrm{PG}(2n, \mathbb{K}')$, $H(2n, \mathbb{K}', \theta)$ has equation $X_0^{\theta+1} + (X_1 X_2^\theta + X_2 X_1^\theta) + \cdots + (X_{2n-1} X_{2n}^\theta + X_{2n} X_{2n-1}^\theta) = 0$.

A *hyperplane* of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a proper subspace meeting each line. A *full (projective) embedding* of $\mathcal{S}$ is an injective mapping e from $\mathcal{P}$ to the point-set of a projective space Σ satisfying (i) $\langle e(\mathcal{P}) \rangle = \Sigma$; (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. If e is a full embedding of $\mathcal{S}$ and if π is a hyperplane of Σ, then $e^{-1}(e(\mathcal{P}) \cap \pi)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(\mathcal{P}) \cap \pi)$ *arises from the embedding* e. Let $Q^-(2n+1, \mathbb{K})$ and $Q^+(2n+1, \mathbb{K}')$ be the quadrics as defined above and let $HS(2n+1, \mathbb{K}')$ denote one of the half-spin geometries for $Q^+(2n+1, \mathbb{K}')$ (as defined in the Introduction). The geometry $HS(2n+1, \mathbb{K}')$ has a nice full embedding into $\mathrm{PG}(2^n-1, \mathbb{K}')$, see Chevalley [5] or Buekenhout and Cameron [2]. We refer to this particular embedding as the *spin embedding* of $HS(2n+1, \mathbb{K}')$. Taking in mind the natural embedding of $DQ^-(2n+1, \mathbb{K})$ into $HS(2n+1, \mathbb{K}')$, we see that the spin embedding of $HS(2n+1, \mathbb{K}')$ induces a full embedding of $DQ^-(2n+1, \mathbb{K})$ into a subspace Σ of $\mathrm{PG}(2^n-1, \mathbb{K}')$. It can be shown, see Cooperstein and Shult [6] and De Bruyn [9] that $\Sigma = \mathrm{PG}(2^n-1, \mathbb{K}')$. The induced embedding of $DQ^-(2n+1, \mathbb{K})$ into $\mathrm{PG}(2^n-1, \mathbb{K}')$ is called the *spin embedding* of $DQ^-(2n+1, \mathbb{K})$.

Let Δ be a thick dual polar space of rank $2n$. A set X of points of Δ is called an *SDPS-set* of Δ if it satisfies the following properties:

(SDPS1) No two points of X are collinear in Δ.

(SDPS2) If $x, y \in X$ such that $\mathrm{d}(x, y) = 2$, then $X \cap \langle x, y \rangle$ is an ovoid of the quad $\langle x, y \rangle$.

(SDPS3) The point-line geometry $\widetilde{\Delta}$ whose points are the elements of X and whose lines are the quads of Δ containing at least two points of X (natural incidence) is a dual polar space of rank n.

(SDPS4) For all $x, y \in X$, $\mathrm{d}(x, y) = 2 \cdot \tilde{\mathrm{d}}(x, y)$, where $\tilde{\mathrm{d}}(x, y)$ denotes the distance between x and y in the dual polar space $\widetilde{\Delta}$.

(SDPS5) If $x \in X$ and L is a line of Δ through x, then L is contained in a (necessarily unique) quad of Δ which contains at least two points of X.

SDPS-sets of dual polar spaces were introduced in De Bruyn and Vandecasteele [11]. The discussion in [11] is however restricted to the finite case. For a discussion including the infinite case, see De Bruyn [7, Section 5.6.7]. SDPS-sets give rise to valuations:

Proposition 2.2 (Theorem 5.29 of [7]) *Let X be an SDPS-set of a thick dual polar space Δ of rank $2n$. For every point x of Δ, we define $f(x) := d(x, X)$. Then f is a valuation of Δ whose maximal value is equal to n.*

A valuation which can be obtained from an SDPS-set in the way as described in Proposition 2.2 is called an *SDPS-valuation*. By Property **(V2)** in the definition of valuation, we have

Corollary 2.3 *Let X be an SDPS-set of a thick dual polar space of rank $2n$. Let H denote the set of points of Δ at distance at most $n-1$ from X. Then H is a hyperplane of Δ (the so-called hyperplane of Δ associated to X).*

SDPS-valuations can be characterized as follows.

Proposition 2.4 (Theorem 5.32 of [7]) *Let Δ be a thick dual polar space and let f be a valuation of Δ with the property that every induced hex valuation is either classical or the extension of an ovoidal valuation of a quad. Then f is the (possibly trivial) extension of an SDPS-valuation of a convex subpolygon of Δ.*

3 Notations and basic lemmas

Let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$. Let θ denote the unique nontrivial element in $Gal(\mathbb{K}'/\mathbb{K})$ and let $n \in \mathbb{N} \setminus \{0, 1\}$.

Let $V(2n+2, \mathbb{K}')$ denote a $(2n+2)$-dimensional vector space over the field $\mathbb{K}'$ and suppose $\mathcal{B}^* = \{\bar{e}_0^*, \bar{e}_1^*, \ldots, \bar{e}_{2n+1}^*\}$ is a basis of $V(2n+2, \mathbb{K}')$. The set of all $\mathbb{K}$-linear combinations of elements of $\mathcal{B}^*$ defines a $(2n+2)$-dimensional vector space $V(2n+2, \mathbb{K})$ over the field $\mathbb{K}$. If $\bar{x} = \sum_{i=0}^{2n+1} X_i \bar{e}_i^*$ is a vector of $V(2n+2, \mathbb{K}')$, then we define $\bar{x}^\theta = \sum_{i=0}^{2n+1} X_i^\theta \bar{e}_i^*$.

Let $\mathrm{PG}(2n+1, \mathbb{K}')$ and $\mathrm{PG}(2n+1, \mathbb{K})$ denote the projective spaces associated to $V(2n+2, \mathbb{K}')$ and $V(2n+2, \mathbb{K})$, respectively. An ordered basis $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ of $V(2n+2, \mathbb{K}')$ is called a *reference system* for $\mathrm{PG}(2n+1, \mathbb{K})$ if $\langle \sum_{i=0}^{2n+1} X_i \bar{e}_i \rangle \in \mathrm{PG}(2n+1, \mathbb{K})$ for all $X_0, X_1, \ldots, X_{2n+1} \in \mathbb{K}$ with $(X_0, X_1, \ldots, X_{2n+1}) \neq (0, 0, \ldots, 0)$. If $p = \langle \sum_{i=0}^{2n+1} X_i \bar{e}_i^* \rangle$ is a point of $\mathrm{PG}(2n+1, \mathbb{K}')$, then we define $p^\theta := \langle \sum_{i=0}^{2n+1} X_i^\theta \bar{e}_i^* \rangle$. For every subspace α of $\mathrm{PG}(2n+1, \mathbb{K}')$, we define $\alpha^\theta := \{p^\theta \,|\, p \in \alpha\}$. Notice that we have given different meanings to the map θ, but from the context it will always be clear what is meant.

There is a natural inclusion of the projective space $\mathrm{PG}(2n+1, \mathbb{K})$ into the projective space $\mathrm{PG}(2n+1, \mathbb{K}')$. In the sequel, we will regard points of $\mathrm{PG}(2n+1, \mathbb{K})$ as points of $\mathrm{PG}(2n+1, \mathbb{K}')$. Every subspace α of $\mathrm{PG}(2n+1, \mathbb{K})$ then generates a subspace α' of $\mathrm{PG}(2n+1, \mathbb{K}')$ of the same dimension as α.

Lemma 3.1 (Lemma 2.1 of [9]) *If α is a subspace of $\mathrm{PG}(2n+1, \mathbb{K}')$, then there exists a unique subspace β of $\mathrm{PG}(2n+1, \mathbb{K})$ such that $\alpha \cap \alpha^\theta = \beta'$.*

For all $i, j \in \{0, \ldots, 2n+1\}$ with $i \leq j$, let $a_{ij} \in \mathbb{K}$ such that

$$q\Big(\sum_{i=0}^{2n+1} X_i \bar{e}_i^*\Big) \quad := \quad \sum_{0 \leq i \leq j \leq 2n+1} a_{ij} X_i X_j$$

is a quadratic form of $V(2n+2, \mathbb{K})$ and $V(2n+2, \mathbb{K}')$ defining a nonsingular quadric $Q^-(2n+1, \mathbb{K})$ of Witt index n in $\mathrm{PG}(2n+1, \mathbb{K})$ and a nonsingular quadric $Q^+(2n+1, \mathbb{K}')$ of Witt index $n+1$ in $\mathrm{PG}(2n+1, \mathbb{K}')$. Let $B(\cdot, \cdot)$ denote the bilinear form of $V(2n+2, \mathbb{K}')$ associated to the quadratic form $q(\cdot)$, i.e.

$$B(\bar{x}_1, \bar{x}_2) \quad = \quad q(\bar{x}_1 + \bar{x}_2) - q(\bar{x}_1) - q(\bar{x}_2)$$

for all $\bar{x}_1, \bar{x}_2 \in V(2n+2, \mathbb{K}')$. Obviously, we have

$$\begin{aligned} q(\bar{x}_1^\theta) &= [q(\bar{x}_1)]^\theta, \\ B(\bar{x}_1^\theta, \bar{x}_2^\theta) &= [B(\bar{x}_1, \bar{x}_2)]^\theta, \end{aligned}$$

for all $\bar{x}_1, \bar{x}_2 \in V(2n+2, \mathbb{K}')$.

Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two systems of generators of $Q^+(2n+1, \mathbb{K}')$ and put $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$.

Lemma 3.2 (Lemma 2.2 of [9]) *We have $(\mathcal{M}^+)^\theta = \mathcal{M}^-$ and $(\mathcal{M}^-)^\theta = \mathcal{M}^+$. As a consequence, for every $\alpha \in \mathcal{M}$, $n - \dim(\alpha \cap \alpha^\theta)$ is odd.*

Lemma 3.3 *Let $k \in \{-1, 0, \ldots, n-1\}$ such that $n-k$ is odd. Then there exists an $\alpha \in \mathcal{M}$ such that* $\dim(\alpha \cap \alpha^\theta) = k$.

Proof. We can choose a reference system $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ for $\mathrm{PG}(2n+1, \mathbb{K})$ and a $\delta \in \mathbb{K}' \setminus \mathbb{K}$ in such a way that a point $\langle \sum_{i=0}^{2n+1} X_i \bar{e}_i \rangle$ of $\mathrm{PG}(2n+1, \mathbb{K})$ belongs to $Q^-(2n+1, \mathbb{K})$ if and only if

$$X_0^2 + (\delta + \delta^\theta) X_0 X_1 + \delta^{\theta+1} X_1^2 + X_2 X_3 + \cdots + X_{2n} X_{2n+1} = 0.$$

Now, let α be the element of $\mathcal{M}$ generated by the points $\langle \delta \bar{e}_0 - \bar{e}_1 \rangle$, $\langle \bar{e}_{4i-2} + \delta \bar{e}_{4i} \rangle$ $(i \in \{1, \ldots, \frac{n-k-1}{2}\})$, $\langle \bar{e}_{4i-1} - \frac{1}{\delta} \bar{e}_{4i+1} \rangle$ $(i \in \{1, \ldots, \frac{n-k-1}{2}\})$, $\langle \bar{e}_{2n-2i} \rangle$ $(i \in \{0, \ldots, k\})$. Then one readily verifies that $\alpha \cap \alpha^\theta = \langle \bar{e}_{2n-2i} \,|\, 0 \leq i \leq k \rangle$. Hence, $\dim(\alpha \cap \alpha^\theta) = k$. ∎

Remark. Let π be a subspace of dimension $k \in \{-1, 0, \ldots, n-3\}$ of $Q^-(2n+1, \mathbb{K})$. The subspaces of $Q^-(2n+1, \mathbb{K})$ through π define a polar space P. The subspaces of $Q^+(2n+1, \mathbb{K}')$ through π' define a polar space P'. We can choose a $\delta \in \mathbb{K}' \setminus \mathbb{K}$ and a reference system $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ for $\mathrm{PG}(2n+1, \mathbb{K})$

such that (i) $q\Big(\sum_{i=0}^{2n+1} X_i\bar{e}_i\Big) = X_0^2 + (\delta + \delta^\theta)X_0X_1 + \delta^{\theta+1}X_1^2 + X_2X_3 + \cdots + X_{2n}X_{2n+1}$, (ii) π is the subspace of $\mathrm{PG}(2n+1,\mathbb{K})$ corresponding to the subspace of $V(2n+2,\mathbb{K})$ generated by $\bar{e}_{2n+1}, \bar{e}_{2n-1}, \ldots, \bar{e}_{2n+1-2k}$. Let $\widetilde{\pi}$ denote the subspace of $\mathrm{PG}(2n+1,\mathbb{K})$ defined by the vectors $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n-2k-1}$. The quadratic form $\widetilde{q}\Big(\sum_{i=0}^{2n-2k-1} X_i\bar{e}_i\Big) = X_0^2 + (\delta+\delta^\theta)X_0X_1 + \delta^{\theta+1}X_1^2 + X_2X_3 + \cdots + X_{2n-2k-2}X_{2n-2k-1}$ defines a nonsingular quadric $\widetilde{Q}^-(2n-2k-1,\mathbb{K})$ of Witt index $n-k-1$ in $\widetilde{\pi}$ and a nonsingular quadric $\widetilde{Q}^+(2n-2k-1,\mathbb{K}')$ of Witt index $n-k$ in $\widetilde{\pi}'$. There exists a natural bijection between the singular subspaces of P (respectively P') and the subspaces contained in the quadric $\widetilde{Q}^-(2n-2k-1,\mathbb{K})$ (respectively $\widetilde{Q}^+(2n-2k-1,\mathbb{K}')$): if α (respectively α') is a subspace of $Q^-(2n+1,\mathbb{K})$ (respectively $Q^+(2n+1,\mathbb{K}')$) through π (respectively π'), then $\alpha \cap \widetilde{\pi}$ (respectively $\alpha' \cap \widetilde{\pi}'$) is a subspace of $\widetilde{Q}^-(2n-2k-1,\mathbb{K})$ (respectively $\widetilde{Q}^+(2n-2k-1,\mathbb{K}')$). Hence, $P \cong Q^-(2n-2k-1,\mathbb{K})$ and $P' \cong Q^+(2n-2k-1,\mathbb{K}')$. Notice also that the elements of one system of generators of $Q^+(2n+1,\mathbb{K}')$ through π' define one system of generators of $P' \cong \widetilde{Q}^+(2n-2k-1,\mathbb{K}')$. We will freely make use of this remark in the sequel.

4 Geometric SDPS-sets of $DQ^-(2n+1,\mathbb{K})$

We will continue with the notation introduced in Section 3. In this section however, we will assume that n is even and that α is an element of $\mathcal{M}$ satisfying $\alpha \cap \alpha^\theta = \emptyset$. By Lemma 3.3 we know that such an α exists. Notice that also $\alpha^\theta \in \mathcal{M}$ and $\alpha \cap \mathrm{PG}(2n+1,\mathbb{K}) = \emptyset$ since every point of $\alpha \cap \mathrm{PG}(2n+1,\mathbb{K})$ is contained in $\alpha \cap \alpha^\theta$.

Lemma 4.1 *For every subspace β of α, $\gamma = \langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$ is a subspace of $\mathrm{PG}(2n+1,\mathbb{K})$ of dimension $2\cdot\dim(\beta)+1$. Moreover, $\gamma' = \langle \beta, \beta^\theta \rangle$.*

Proof. Since $\beta \subseteq \alpha$ and $\beta^\theta \subseteq \alpha^\theta$ are disjoint, $\langle \beta, \beta^\theta \rangle$ has dimension $2 \cdot \dim(\beta) + 1$. Now, by Lemma 3.1, there exists a subspace γ_1 of $\mathrm{PG}(2n+1,\mathbb{K})$ such that $\gamma_1' = \langle \beta, \beta^\theta \rangle \cap \langle \beta, \beta^\theta \rangle^\theta = \langle \beta, \beta^\theta \rangle$. Obviously, $\dim(\gamma_1) = \dim(\langle \beta, \beta^\theta \rangle) = 2 \cdot \dim(\beta) + 1$ and $\gamma_1 = \langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$. ■

Now, let H denote the set of all points $\langle \bar{x} \rangle$ of α for which $h(\bar{x}) := B(\bar{x}, \bar{x}^\theta) = 0$. Obviously, H is a θ-Hermitian variety of α. We observe the following for two points $\langle \bar{x} \rangle$, $\langle \bar{y} \rangle$ of α:

(I) $\langle \bar{x} \rangle$ and $\langle \bar{y}^\theta \rangle$ are collinear on the quadric $Q^+(2n+1,\mathbb{K}')$ if and only if $B(\bar{x}, \bar{y}^\theta) = 0$;

(II) if $\langle \bar{x} \rangle \in H$ and $\langle \bar{y} \rangle \neq \langle \bar{x} \rangle$, then $B(\bar{x}, \bar{y}^\theta) = 0$ if and only if the line of α through $\langle \bar{x} \rangle$ and $\langle \bar{y} \rangle$ is either contained in H or intersects H in the point $\langle \bar{x} \rangle$.

By (I), a point $p \in \alpha$ belongs to H if and only if p is collinear on $Q^+(2n+1, \mathbb{K}')$ with p^θ.

Lemma 4.2 *H is nonsingular.*

Proof. Suppose $\langle \bar{x} \rangle$ is a singular point of H. Then by (II) above, $B(\bar{x}, \bar{y}^\theta) = 0$ for all $\bar{y} \in V(2n+2, \mathbb{K}')$ such that $\langle \bar{y} \rangle$ is a point of α. Hence, by (I) above, $\langle \bar{x} \rangle$ is collinear on $Q^+(2n+1, \mathbb{K}')$ with every point of α^θ. This is impossible since α^θ is a generator of $Q^+(2n+1, \mathbb{K}')$ and $\langle \bar{x} \rangle \notin \alpha^\theta$. ■

Lemma 4.3 *If β is a subspace of α contained in H, then $\langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K})$ is a subspace of $Q^-(2n+1, \mathbb{K})$ of dimension $2 \cdot \dim(\beta) + 1$.*

Proof. Put $k := \dim(\beta) + 1$ and let $\{p_1, p_2, \dots, p_k\}$ be an independent generating set of points for the subspace β. Then $\{p_1, p_2, \dots, p_k, p_1^\theta, p_2^\theta, \dots, p_k^\theta\}$ is an independent generating set of points for the subspace $\langle \beta, \beta^\theta \rangle$. Now, by (I) and (II) above, $\{p_1, p_2, \dots, p_k, p_1^\theta, p_2^\theta, \dots, p_k^\theta\}$ is a set of mutually collinear points of the quadric $Q^+(2n+1, \mathbb{K}')$. By Lemma 4.1, it now follows that $\langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K})$ is a subspace of dimension $2 \cdot \dim(\beta) + 1$ of $Q^-(2n+1, \mathbb{K})$. ■

Lemma 4.4 *Let x be a point of $\mathrm{PG}(2n+1, \mathbb{K})$. Then there exists a unique line L_x in $\mathrm{PG}(2n+1, \mathbb{K}')$ through x which meets α and α^θ in points. Moreover, $(L_x \cap \alpha)^\theta = L_x \cap \alpha^\theta$ and $L_x \cap \mathrm{PG}(2n+1, \mathbb{K})$ is a line of $\mathrm{PG}(2n+1, \mathbb{K})$. If $x \in Q^-(2n+1, \mathbb{K})$, then $L_x \subseteq Q^+(2n+1, \mathbb{K}')$ and $L_x \cap \mathrm{PG}(2n+1, \mathbb{K})$ is a line of $Q^-(2n+1, \mathbb{K})$.*

Proof. Clearly, there is a unique line L_x through x meeting α and α^θ in points, namely the line through the points $\langle \alpha, x \rangle \cap \alpha^\theta$ and $\langle \alpha^\theta, x \rangle \cap \alpha$. Since L_x meets α and α^θ and contains the point x, also L_x^θ meets α and α^θ and contains the point $x^\theta = x$. Hence, $L_x^\theta = L_x$. This implies that $(L_x \cap \alpha)^\theta = L_x \cap \alpha^\theta$. By Lemma 4.1, $L_x \cap \mathrm{PG}(2n+1, \mathbb{K})$ is a line of $\mathrm{PG}(2n+1, \mathbb{K})$.

Suppose now that $x \in Q^-(2n+1, \mathbb{K})$. Then the line L_x contains three points of $Q^+(2n+1, \mathbb{K}')$, namely the point x and the unique points in $L_x \cap \alpha$ and $L_x \cap \alpha^\theta$. Hence, $L_x \subseteq Q^+(2n+1, \mathbb{K}')$. It follows that $L_x \cap \mathrm{PG}(2n+1, \mathbb{K})$ is a line of $Q^-(2n+1, \mathbb{K})$. ■

Lemma 4.5 *Let β be a subspace of α contained in H and let γ be the subspace $\langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K})$ of $Q^-(2n+1, \mathbb{K})$. Let x be a point of*

$Q^-(2n+1,\mathbb{K}) \setminus \gamma$ which is collinear on $Q^-(2n+1,\mathbb{K})$ with every point of γ, and let L_x denote the unique line of $\mathrm{PG}(2n+1,\mathbb{K}')$ through x which meets α and α^θ in the respective points v and v^θ. Then

(i) L_x and $\langle \beta, \beta^\theta \rangle$ are disjoint;

(ii) the subspace $\langle \beta, v \rangle$ of α is contained in H.

Proof. (i) Since $x \notin \gamma$, also $x \notin \langle \beta, \beta^\theta \rangle$. Suppose $L_x \cap \langle \beta, \beta^\theta \rangle$ is a singleton $\{y\}$. By Lemma 4.4, L_x is generated by a line of $\mathrm{PG}(2n+1,\mathbb{K})$ which is contained in $Q^-(2n+1,\mathbb{K})$. Since both L_x and $\langle \beta, \beta^\theta \rangle = \gamma'$ are generated by subspaces of $\mathrm{PG}(2n+1,\mathbb{K})$, the point y must belong to $\mathrm{PG}(2n+1,\mathbb{K})$. Since $y \in \langle \beta, \beta^\theta \rangle \setminus (\beta \cup \beta^\theta)$, there exists a unique line through y meeting β and β^θ and this line necessarily coincides with the unique line through y meeting α and α^θ. It follows that L_x meets β and β^θ, contradicting the fact that L_x is not contained in $\langle \beta, \beta^\theta \rangle$ (recall $x \notin \langle \beta, \beta^\theta \rangle$). Hence, L_x and $\langle \beta, \beta^\theta \rangle$ are disjoint.

(ii) We have $\beta \subseteq H$. Since $L_x \subseteq Q^+(2n+1,\mathbb{K}')$, v and v^θ are collinear on $Q^+(2n+1,\mathbb{K}')$, i.e. $v \in H$. In order to show that $\langle \beta, v \rangle \subseteq H$, we need to prove that every point u of β is collinear on H with v, or equivalently, that every point u of β is collinear with v^θ on the quadric $Q^+(2n+1,\mathbb{K}')$ (see (I) and (II) above).

Since x is collinear on $Q^-(2n+1,\mathbb{K})$ with every point of γ, it is collinear on $Q^+(2n+1,\mathbb{K}')$ with every point of $\gamma' = \langle \beta, \beta^\theta \rangle$. In particular, x is collinear on $Q^+(2n+1,\mathbb{K}')$ with u. Now, since u is collinear on $Q^+(2n+1,\mathbb{K}')$ with v and x, it is also collinear on $Q^+(2n+1,\mathbb{K}')$ with v^θ. This is precisely what we needed to show. ■

Proposition 4.6 *H is a nonsingular θ-Hermitian variety of (maximal) Witt index $\frac{n}{2}$ in α.*

Proof. In view of Lemma 4.2, we need to show that there exists an $(\frac{n}{2}-1)$-dimensional subspace on H.

We prove by induction on $k \in \{0, \ldots, \frac{n}{2}\}$ that there exists a subspace β_k of dimension $k-1$ on H. Obviously, this claim holds if $k=0$. So, suppose $k \geq 1$. By the induction hypothesis, there exists a subspace β_{k-1} of dimension $k-2$ on H. Put $\gamma_{k-1} := \langle \beta_{k-1}, \beta_{k-1}^\theta \rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$. By Lemma 4.3, γ_{k-1} is a subspace of dimension $2k-3$ of $Q^-(2n+1,\mathbb{K})$. Since $k \leq \frac{n}{2}$, there exists a point $u_k \in Q^-(2n+1,\mathbb{K})$ which is collinear on $Q^-(2n+1,\mathbb{K})$ with every point of γ_{k-1}. Let L_{u_k} denote the unique line through u_k meeting α and α^θ in the respective points v_k and ${v_k}^\theta$ (see Lemma 4.4). By Lemma 4.5, $\beta_k := \langle \beta_{k-1}, v_k \rangle \subseteq H$ and $\dim(\beta_k) = k-1$. ■

Proposition 4.7 *Let X be the set of generators of $Q^-(2n+1,\mathbb{K})$ of the form $\langle\beta,\beta^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$, where β is some generator of H. Then X is an SDPS-set of the dual polar space $DQ^-(2n+1,\mathbb{K})$. Moreover, the dual polar space defined on the set X by the quads of $DQ^-(2n+1,\mathbb{K})$ containing at least two points of X is isomorphic to the dual polar space associated to H.*

Proof. Let $\mathrm{d}(\cdot,\cdot)$ denote the distance function in the dual polar space $DQ^-(2n+1,\mathbb{K})$. Let $DH(n,\mathbb{K}',\theta)$ denote the dual polar space associated to $H = H(n,\mathbb{K}',\theta)$ and let $\mathrm{d}'(\cdot,\cdot)$ denote the distance function in $DH(n,\mathbb{K}',\theta)$.

For every subspace γ of H, we define $\gamma^\phi := \langle\gamma,\gamma^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$. By Lemma 4.3, γ^ϕ is a subspace of $Q^-(2n+1,\mathbb{K})$ of dimension $2\cdot\dim(\gamma)+1$. So, if γ is a point of $DH(n,\mathbb{K}',\theta)$, then γ^ϕ is a point of $DQ^-(2n+1,\mathbb{K})$. If γ_1 and γ_2 are two distinct subspaces on H, then $\gamma_1^\phi \cap \gamma_2^\phi = \langle\gamma_1,\gamma_1^\phi\rangle \cap \langle\gamma_2,\gamma_2^\phi\rangle \cap \mathrm{PG}(2n+1,\mathbb{K}) = \langle\gamma_1\cap\gamma_2,(\gamma_1\cap\gamma_2)^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K}) = (\gamma_1\cap\gamma_2)^\phi$. Hence,

$$\mathrm{d}(\beta_1^\phi,\beta_2^\phi) = 2\cdot\mathrm{d}'(\beta_1,\beta_2) \tag{1}$$

for any two points β_1 and β_2 of $DH(n,\mathbb{K}',\theta)$. This proves property **(SDPS1)**. It is also obvious that ϕ defines a bijection between the set of lines of $DH(n,\mathbb{K}',\theta)$ and the set of quads of $DQ^-(2n+1,\mathbb{K})$ which contain at least two points of X. As a consequence, the partial linear space $\widetilde{\Delta}$ whose points are the elements of X and whose lines are the quads of $DQ^-(2n+1,\mathbb{K})$ containing at least two points of X (natural incidence) is isomorphic to $DH(n,\mathbb{K}',\theta)$, proving property **(SDPS3)**. Property **(SDPS4)** now immediately follows from equation (1).

We now prove property **(SDPS2)**. Let β_1 be a line of $DH(n,\mathbb{K}',\theta)$ and put $\gamma_1 := \langle\beta_1,\beta_1^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K}) \subseteq Q^-(2n+1,\mathbb{K})$. Let γ_2 be an arbitrary subspace of dimension $n-2$ of $Q^-(2n+1,\mathbb{K})$ containing γ_1. We need to prove that there exists a unique generator β_2 of $H(n,\mathbb{K}',\theta)$ through β_1 such that $\gamma_2 \subseteq \langle\beta_2,\beta_2^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$. Let x be an arbitrary point of $\gamma_2\setminus\gamma_1$ and let L_x denote the unique line through x meeting α and α^θ in the respective points v and v^θ. By Lemma 4.5, $L_x \cap \langle\beta_1,\beta_1^\theta\rangle = \emptyset$ and v is collinear on the Hermitian variety H with every point of β_1. If we put $\beta^* := \langle\beta_1, v\rangle$, then β^* is a generator of $H(n,\mathbb{K}',\theta)$ through β_1 satisfying $\gamma_2 \subseteq \langle\beta^*,(\beta^*)^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$. Conversely, suppose that β_2 is a generator of $H(n,\mathbb{K}',\theta)$ through β_1 such that $\gamma_2 \subseteq \langle\beta_2,\beta_2^\theta\rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$. Since $x \in \langle\beta_2,\beta_2^\theta\rangle \setminus (\beta_2 \cup \beta_2^\theta)$, there exists a unique line through x meeting β_2 and β_2^θ. This line necessarily coincides with L_x. Hence, $v \in \beta_2$ and $\beta_2 = \langle\beta_1, v\rangle = \beta^*$. Property **(SDPS2)** immediately follows.

We now prove property **(SDPS5)**. Let γ_1 be a generator of $Q^-(2n+1,\mathbb{K})$ corresponding to a point of X and let γ_2 be an arbitrary hyperplane of γ_1.

There exists a unique generator β_1 of H such that $\langle \beta_1, \beta_1^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K}) = \gamma_1$. Now, γ_2' is a hyperplane of $\gamma_1' = \langle \beta_1, \beta_1^\theta \rangle$ and hence intersects β_1 in either β_1 or a hyperplane of β_1. Suppose $\beta_1 \subseteq \gamma_2'$. Then $\beta_1^\theta \subseteq {\gamma_2'}^\theta = \gamma_2'$ and hence $\langle \beta_1, \beta_1^\theta \rangle \subseteq \gamma_2'$, a contradiction. Hence, γ_2' intersects β_1 in a hyperplane β_2 of β_1. Since $\beta_2 \subseteq \gamma_2'$, we have $\beta_2^\theta \subseteq {\gamma_2'}^\theta = \gamma_2'$, $\langle \beta_2, \beta_2^\theta \rangle \subseteq \gamma_2'$ and hence $\langle \beta_2, \beta_2^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K}) \subseteq \gamma_2' \cap \mathrm{PG}(2n+1, \mathbb{K}) = \gamma_2$. So, the $(n-3)$-dimensional subspace $\langle \beta_2, \beta_2^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K})$ of $Q^-(2n+1, \mathbb{K})$ corresponds to a quad of $DQ^-(2n+1, \mathbb{K})$ which contains the line of $DQ^-(2n+1, \mathbb{K})$ corresponding to γ_2. This proves property **(SDPS5)**. ■

SDPS-sets of the dual polar space $DQ^-(2n+1, \mathbb{K})$ which can be obtained as described in Proposition 4.7 are called *geometric SDPS-sets* of $DQ^-(2n+1, \mathbb{K})$. A certain class of SDPS-sets of $DQ^-(2n+1, \mathbb{K})$ has already been described in De Bruyn & Vandecasteele [11] and Pralle & Shpectorov [15]. All these SDPS-sets are geometric. We will prove this in the appendix of this paper using the description of [11].

Definition. Again, suppose that n is even and consider the inclusion $\mathrm{PG}(n-1, \mathbb{K}) \subset \mathrm{PG}(n-1, \mathbb{K}')$. We denote by θ here the conjugation in $\mathrm{PG}(n-1, \mathbb{K}')$ with respect to the field extension $\mathbb{K}'/\mathbb{K}$. There exists an $(\frac{n}{2}-1)$-dimensional subspace β of $\mathrm{PG}(n-1, \mathbb{K}')$ such that $\beta \cap \beta^\theta = \emptyset$. For every point $x \in \beta$, $L_x := xx^\theta \cap \mathrm{PG}(n-1, \mathbb{K})$ is a line of $\mathrm{PG}(n-1, \mathbb{K})$. The set $S = \{L_x \,|\, x \in \beta\}$ is a *spread* of $\mathrm{PG}(n-1, \mathbb{K})$, i.e. a set of lines of $\mathrm{PG}(n-1, \mathbb{K})$ partitioning the point-set of $\mathrm{PG}(n-1, \mathbb{K})$. Any spread of $\mathrm{PG}(n-1, \mathbb{K})$ which can be obtained in this way is called a *regular spread*. For a discussion of regular spreads in the finite case, see Hirschfeld [12, Chapter 4] and [13, Chapter 17].

Let X be as defined in Proposition 4.7 and let x be a point of X. The convex subspaces of $DQ^-(2n+1, \mathbb{K})$ containing the point x define a projective space $\mathcal{L}_x$ isomorphic to $\mathrm{PG}(n-1, \mathbb{K})$. The quads through x containing at least two points of X define a spread S_x of $\mathcal{L}_x$ by property **(SDPS5)**.

Proposition 4.8 *For every point x of X, the spread S_x of $\mathcal{L}_x$ is regular.*

Proof. Let γ be the generator of $Q^-(2n+1, \mathbb{K})$ corresponding to x. Then there exists a generator β of H such that $\gamma = \langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K})$. The lines of the spread S_x of $\mathcal{L}_x$ correspond to the subspaces $\langle \eta, \eta^\theta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K}) = \langle \eta, \beta^\theta \rangle \cap \langle \eta^\theta, \beta \rangle \cap \mathrm{PG}(2n+1, \mathbb{K})$, where η is some hyperplane of β. In this way, we obtain a regular spread in the dual projective space associated to γ. This proves the proposition. ■

The following proposition is precisely Theorem 1.2.

Proposition 4.9 *Any two geometric SDPS-sets of $DQ^-(2n+1,\mathbb{K})$ are isomorphic.*

Proof. Let V be the subspace of $V(2n+2,\mathbb{K}')$ whose nonzero elements consist of all vectors $\bar{x}$ for which $\langle\bar{x}\rangle \in \alpha$. For all vectors $\bar{x}$ and $\bar{y}$ of V, we define $H(\bar{x},\bar{y}) := B(\bar{x},\bar{y}^\theta)$. Then $H(\cdot,\cdot)$ is a Hermitian form on V and H is the Hermitian variety of α associated to it. Let δ be an element of $\mathbb{K}'$ such that $\delta^\theta \notin \{\delta,-\delta\}$. [If $char(\mathbb{K})=2$, then δ is an arbitrary element of $\mathbb{K}'\setminus\mathbb{K}$. If $char(\mathbb{K})\neq 2$, then for an arbitrary $\mu\in\mathbb{K}'\setminus\mathbb{K}$, δ can be chosen in the set $\{\mu,\mu+1\}$.] Now, we can always choose a $k\in\mathbb{K}\setminus\{0\}$ and vectors $\bar{f}_0$, $\bar{f}_i$ ($i\in\{1,\ldots,\frac{n}{2}\}$), $\bar{g}_i$ ($i\in\{1,\ldots,\frac{n}{2}\}$) in V such that

- $\alpha = \langle \bar{f}_0,\bar{f}_1,\ldots,\bar{f}_{\frac{n}{2}},\bar{g}_1,\ldots,\bar{g}_{\frac{n}{2}}\rangle$,
- $H(\bar{f}_0,\bar{f}_0) = -k(\delta-\delta^\theta)^2$,
- $H(\bar{f}_0,\bar{f}_i) = H(\bar{f}_0,\bar{g}_i) = 0$ for all $i\in\{1,\ldots,\frac{n}{2}\}$,
- $H(\bar{f}_i,\bar{f}_j) = H(\bar{g}_i,\bar{g}_j) = 0$ for all $i,j\in\{1,\ldots,\frac{n}{2}\}$,
- $H(\bar{f}_i,\bar{g}_i) = k\cdot\frac{\delta^\theta-\delta}{\delta^\theta}$ for every $i\in\{1,\ldots,\frac{n}{2}\}$,
- $H(\bar{f}_i,\bar{g}_j) = 0$ for all $i,j\in\{1,\ldots,\frac{n}{2}\}$ with $i\neq j$.

[If β_1 and β_2 are two disjoint generators of H and $p=\langle\beta_1,\beta_2\rangle^\zeta$, where ζ is the Hermitian polarity of α associated to H, then we can choose $\bar{f}_0,\bar{f}_1,\ldots,\bar{f}_{\frac{n}{2}},\bar{g}_1,\ldots,\bar{g}_{\frac{n}{2}}$ in such a way that $p=\langle\bar{f}_0\rangle$, $\beta_1=\langle\bar{f}_1,\ldots,\bar{f}_{\frac{n}{2}}\rangle$ and $\beta_2=\langle\bar{g}_1,\ldots,\bar{g}_{\frac{n}{2}}\rangle$.] Now, put

$$\bar{e}_0 = \frac{\bar{f}_0^\theta-\bar{f}_0}{\delta^\theta-\delta}, \qquad \bar{e}_1 = \frac{\delta\bar{f}_0^\theta-\delta^\theta\bar{f}_0}{\delta^\theta-\delta},$$

and

$$\bar{e}_{4i-2} = \frac{\delta^\theta\bar{f}_i-\delta\bar{f}_i^\theta}{\delta^\theta-\delta}, \qquad \bar{e}_{4i} = \frac{\bar{f}_i^\theta-\bar{f}_i}{\delta^\theta-\delta},$$

$$\bar{e}_{4i-1} = \frac{\delta^\theta\bar{g}_i^\theta-\delta\bar{g}_i}{\delta^\theta-\delta}, \qquad \bar{e}_{4i+1} = \frac{\delta^{\theta+1}(\bar{g}_i^\theta-\bar{g}_i)}{\delta^\theta-\delta},$$

for every $i\in\{1,\ldots,\frac{n}{2}\}$. Then $\bar{e}_0,\bar{e}_1,\ldots,\bar{e}_{2n+1}\in V(2n+2,\mathbb{K})$. Moreover, these $2n+2$ vectors are linearly independent since $\alpha\cap\alpha^\theta=\emptyset$. Hence, $(\bar{e}_0,\bar{e}_1,\ldots,\bar{e}_{2n+1})$ is a reference system for $\mathrm{PG}(2n+1,\mathbb{K})$. Suppose

$$q(\sum_{i=0}^{2n+1}X_i\bar{e}_i) = \sum_{0\leq i\leq j\leq 2n+1} a_{ij}X_iX_j.$$

Let $i \in \{1, \ldots, \frac{n}{2}\}$. Since $\langle \bar{f}_i \rangle \in H$, $\langle \bar{f}_i \rangle$ and $\langle \bar{f}_i^\theta \rangle$ are collinear points on $Q^+(2n+1, \mathbb{K}')$. Hence, $\langle \bar{e}_{4i-2} \rangle, \langle \bar{e}_{4i} \rangle \in Q^-(2n+1, \mathbb{K})$. In a similar way, one can prove that $\langle \bar{e}_{4i-1} \rangle$, $\langle \bar{e}_{4i+1} \rangle \in Q^-(2n+1, \mathbb{K})$. We can conclude that $a_{ii} = 0$ for every $i \in \{2, \ldots, 2n+1\}$.

Notice that since α is a generator of $Q^+(2n+1, \mathbb{K}')$, $B(\bar{x}, \bar{y}) = H(\bar{x}, \bar{y}^\theta) = 0$ for all $\bar{x}, \bar{y} \in \{\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{\frac{n}{2}}, \bar{g}_1, \ldots, \bar{g}_{\frac{n}{2}}\}$.

We calculate

$$\begin{aligned}
a_{01} &= B(\bar{e}_0, \bar{e}_1) \\
&= B(\frac{\bar{f}_0^\theta - \bar{f}_0}{\delta^\theta - \delta}, \frac{\delta \bar{f}_0^\theta - \delta^\theta \bar{f}_0}{\delta^\theta - \delta}) \\
&= \frac{\delta \cdot B(\bar{f}_0^\theta, \bar{f}_0^\theta) - \delta^\theta \cdot B(\bar{f}_0^\theta, \bar{f}_0) - \delta \cdot B(\bar{f}_0, \bar{f}_0^\theta) + \delta^\theta \cdot B(\bar{f}_0, \bar{f}_0)}{(\delta^\theta - \delta)^2}.
\end{aligned}$$

Now, $B(\bar{f}_0, \bar{f}_0) = 0$, $B(\bar{f}_0^\theta, \bar{f}_0^\theta) = (B(\bar{f}_0, \bar{f}_0))^\theta = 0$ and $B(\bar{f}_0^\theta, \bar{f}_0) = B(\bar{f}_0, \bar{f}_0^\theta) = H(\bar{f}_0, \bar{f}_0) = -k(\delta - \delta^\theta)^2$. It follows that

$$a_{01} = k(\delta + \delta^\theta).$$

After some straightforward calculations, one finds in a similar way that

- $a_{0i} = a_{1i} = 0$ for all $i \in \{2, \ldots, 2n+1\}$,
- $a_{2i,2i+1} = k$ for all $i \in \{1, \ldots, n\}$,
- $a_{j_1,j_2} = 0$ for all $j_1, j_2 \in \{2, \ldots, 2n+1\}$ with $j_1 < j_2$ and (j_1, j_2) not of the form $(2i, 2i+1)$ for some $i \in \{1, \ldots, n\}$.

Now, since $\langle \bar{f}_0 \rangle = \langle \delta \bar{e}_0 - \bar{e}_1 \rangle \in \alpha$ and $\langle \bar{f}_0^\theta \rangle = \langle \delta^\theta \bar{e}_0 - \bar{e}_1 \rangle \in \alpha^\theta$ are points of $Q^+(2n+1, \mathbb{K}')$, we have

$$\begin{cases} a_{00} \cdot \delta^2 + k(\delta + \delta^\theta)(-\delta) + a_{11} = 0, \\ a_{00} \cdot (\delta^\theta)^2 + k(\delta + \delta^\theta)(-\delta^\theta) + a_{11} = 0. \end{cases}$$

Hence, $a_{00} = k$ and $a_{11} = k\delta^{\theta+1}$. So, with respect to the reference system $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ of $\mathrm{PG}(2n+1, \mathbb{K})$, $Q^-(2n+1, \mathbb{K})$ has equation

$$X_0^2 + (\delta + \delta^\theta) X_0 X_1 + \delta^{\theta+1} X_1^2 + X_2 X_3 + \cdots + X_{2n} X_{2n+1} = 0.$$

Now, suppose $\alpha^\dagger$ is another generator of $Q^+(2n+1, \mathbb{K}')$ such that $(\alpha^\dagger)^\theta \cap \alpha^\dagger = \emptyset$. Then construct in the same way as above a reference system $(\bar{e}_0^\dagger, \bar{e}_1^\dagger, \ldots, \bar{e}_{2n+1}^\dagger)$ for $\mathrm{PG}(2n+1, \mathbb{K})$ associated to suitable vectors $\bar{f}_0^\dagger$, $\bar{f}_i^\dagger$ $(i \in \{1, \ldots, \frac{n}{2}\})$, $\bar{g}_i^\dagger$ $(i \in \{1, \ldots, \frac{n}{2}\})$. With respect to the reference system

$(\bar{e}_0^\dagger, \bar{e}_1^\dagger, \ldots, \bar{e}_{2n+1}^\dagger)$ of $\mathrm{PG}(2n+1, \mathbb{K})$, $Q^-(2n+1, \mathbb{K})$ has also equation $X_0^2 + (\delta + \delta^\theta) X_0 X_1 + \delta^{\theta+1} X_1^2 + X_2 X_3 + \cdots + X_{2n} X_{2n+1} = 0$. It is now clear that the linear map $\sum_{i=0}^{2n+1} X_i \bar{e}_i \mapsto \sum_{i=0}^{2n+1} X_i \bar{e}_i^\dagger$ of $V(2n+2, \mathbb{K}')$ induces an automorphism of $\mathrm{PG}(2n+1, \mathbb{K}')$ fixing $\mathrm{PG}(2n+1, \mathbb{K})$ and $Q^-(2n+1, \mathbb{K})$ setwise and mapping $\alpha = \langle \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{\frac{n}{2}}, \bar{g}_1, \ldots, \bar{g}_{\frac{n}{2}} \rangle$ to $\alpha^\dagger = \langle \bar{f}_0^\dagger, \bar{f}_1^\dagger, \ldots, \bar{f}_{\frac{n}{2}}^\dagger, \bar{g}_1^\dagger, \ldots, \bar{g}_{\frac{n}{2}}^\dagger \rangle$. Hence, the geometric SDPS-sets of $DQ^-(2n+1, \mathbb{K})$ associated to α and $\alpha^\dagger$ are isomorphic. ∎

5 The natural embedding of $DQ^-(2n+1, \mathbb{K})$ into the half-spin geometry for $Q^+(2n+1, \mathbb{K}')$

We will continue with the notation introduced in Section 3. For every $\alpha \in \mathcal{M}$ and every generator γ of $Q^-(2n+1, \mathbb{K})$, we define

$$f_\alpha(\gamma) \;\; := \;\; M - \dim(\gamma' \cap \alpha),$$

where

$$M \;\; := \;\; \max\{\dim(\eta' \cap \alpha) \,|\, \eta \text{ is a generator of } Q^-(2n+1, \mathbb{K})\}.$$

Proposition 5.1 *For every $\alpha \in \mathcal{M}$, f_α is a valuation of the dual polar space $DQ^-(2n+1, \mathbb{K})$ associated to $Q^-(2n+1, \mathbb{K})$.*

Proof. By definition, the minimal value attained by f_α is equal to 0. So, property **(V1)** is satisfied.

Let β be an arbitrary $(n-2)$-dimensional subspace of $Q^-(2n+1, \mathbb{K})$. Then there exists a unique generator η of $Q^+(2n+1, \mathbb{K}')$ through β' for which $\dim(\alpha \cap \eta) = \dim(\alpha \cap \beta') + 2$. Let γ be the unique subspace of $\mathrm{PG}(2n+1, \mathbb{K})$ such that $\gamma' = \eta \cap \eta^\theta$ (see Lemma 3.1). Then $\gamma \subseteq Q^-(2n+1, \mathbb{K})$ and $\beta' \subseteq \gamma' \subseteq \eta$. By Lemma 3.2, $\dim(\gamma') = n-1$. So, γ is a generator of $Q^-(2n+1, \mathbb{K})$ through β. Since $\beta' \subset \gamma' \subset \eta$ and $\dim(\alpha \cap \eta) = \dim(\alpha \cap \beta') + 2$, $\dim(\alpha \cap \gamma') = \dim(\alpha \cap \beta') + 1$. Conversely, suppose that κ is a generator of $Q^-(2n+1, \mathbb{K})$ through β such that $\dim(\alpha \cap \kappa') = \dim(\alpha \cap \beta') + 1$. Then κ' is necessarily contained in η. Then $\kappa' = {\kappa'}^\theta \subseteq \eta^\theta$ and $\kappa' \subseteq \eta \cap \eta^\theta = \gamma'$. Since κ' and γ' have the same dimension, we have $\kappa = \gamma$. It follows that the line of $DQ^-(2n+1, \mathbb{K})$ corresponding to β has a unique point with smallest f_α-value, namely the point corresponding to γ, and that all the remaining points of that line have value $f_\alpha(\gamma) + 1$. This proves that property **(V2)** is satisfied.

Now, let β be an arbitrary generator of $Q^-(2n+1, \mathbb{K})$. By Lemma 3.1, there exists a subspace γ of $\mathrm{PG}(2n+1, \mathbb{K})$ such that $\gamma' = \langle \alpha \cap \beta', \alpha^\theta \cap \beta' \rangle \cap$

$\langle\alpha\cap\beta',\alpha^\theta\cap\beta'\rangle^\theta = \langle\alpha\cap\beta',\alpha^\theta\cap\beta'\rangle \subseteq \beta'$. Let F_β denote the convex subspace of $DQ^-(2n+1,\mathbb{K})$ corresponding to the subspace γ of $Q^-(2n+1,\mathbb{K})$. Obviously, the point of $DQ^-(2n+1,\mathbb{K})$ corresponding to β belongs to F_β.

We will now prove that property **(V3)** is satisfied with respect to the convex subspace F_β. Let η be a generator of $Q^-(2n+1,\mathbb{K})$ through γ. Then η' contains $\gamma' = \langle\alpha\cap\beta',\alpha^\theta\cap\beta'\rangle$ and hence $\dim(\eta'\cap\alpha) \geq \dim(\alpha\cap\beta')$, i.e. $f_\alpha(\eta) \leq f_\alpha(\beta)$. Now, let κ be an arbitrary generator of $Q^-(2n+1,\mathbb{K})$ such that $f_\alpha(\kappa) = f_\alpha(\eta)-1$ and $\dim(\eta\cap\kappa) = n-2$. So, $\dim(\alpha\cap\kappa') = \dim(\alpha\cap\eta')+1$ and $\dim(\kappa'\cap\eta') = n-2$. Let p be an arbitrary point of $(\alpha\cap\kappa')\setminus(\alpha\cap\eta')$. Then $\kappa'\cap\eta'$ is the set of points of η' collinear with p on $Q^+(2n+1,\mathbb{K}')$. Since every point of $\alpha\cap\eta'$ is collinear on $Q^+(2n+1,\mathbb{K}')$ with $p\in\alpha$, $\alpha\cap\eta'\subseteq\eta'\cap\kappa'$, i.e. $\alpha\cap\eta'\subseteq\kappa'$. Hence, also $\alpha^\theta\cap\eta'\subseteq\kappa'$. Since $\alpha\cap\beta'\subseteq\alpha\cap\eta'$ (recall $\eta'\supseteq\gamma' = \langle\alpha\cap\beta',\alpha^\theta\cap\beta'\rangle$) and $\alpha^\theta\cap\beta'\subseteq\alpha^\theta\cap\eta'$, $\gamma' = \langle\alpha\cap\beta',\alpha^\theta\cap\beta'\rangle \subseteq \langle\alpha\cap\eta',\alpha^\theta\cap\eta'\rangle \subseteq \kappa'$, i.e. $\gamma\subseteq\kappa$. So, f_α satisfies property **(V3)**. ■

Proposition 5.2 *Suppose there exists a generator β of $Q^-(2n+1,\mathbb{K})$ such that $\beta'\subseteq\alpha$. Then f_α is a classical valuation of $DQ^-(2n+1,\mathbb{K})$, namely, for every generator γ of $Q^-(2n+1,\mathbb{K})$, $f_\alpha(\gamma)$ equals the distance $d(\beta,\gamma)$ between β and γ in the dual polar space $DQ^-(2n+1,\mathbb{K})$.*

Proof. From $\beta'\subseteq\alpha$, it follows $\beta' = \beta'^\theta \subseteq \alpha^\theta$ and hence $\beta' = \alpha\cap\alpha^\theta$ (recall Lemma 3.2). Let γ be an arbitrary generator of $Q^-(2n+1,\mathbb{K})$. Suppose γ' contains a point x of $\alpha\setminus\beta'$. Since $x, x^\theta \in \gamma' \subseteq Q^+(2n+1,\mathbb{K}')$, $xx^\theta \subseteq Q^+(2n+1,\mathbb{K}')$ and $\langle\alpha,\alpha^\theta\rangle \subseteq Q^+(2n+1,\mathbb{K}')$. This is impossible since α and α^θ are generators of $Q^+(2n+1,\mathbb{K}')$. Hence, $\gamma'\cap\alpha\subseteq\beta'$, i.e. $\gamma'\cap\alpha = \gamma'\cap\beta'$. Hence, $f_\alpha(\gamma) = M - \dim(\gamma'\cap\alpha) = M - \dim(\gamma'\cap\beta') = M - \dim(\gamma\cap\beta)$, where $M = \max\{\dim(\eta'\cap\alpha)\,|\,\ldots\} = \max\{\dim(\eta'\cap\beta')\,|\,\ldots\} = \max\{\dim(\eta\cap\beta)\,|\,\ldots\} = n-1$. So, $f_\alpha(\gamma)$ equals the distance between β and γ in the dual polar space $DQ^-(2n+1,\mathbb{K})$. ■

Lemma 5.3 *Let x be a point of $\alpha\cap\alpha^\theta\cap\mathrm{PG}(2n+1,\mathbb{K})$, let β be a generator of $Q^-(2n+1,\mathbb{K})$ not containing x and let γ be the unique generator of $Q^-(2n+1,\mathbb{K})$ containing x intersecting β in a subspace of dimension $n-2$. Then* $\dim(\gamma'\cap\alpha) = \dim(\beta'\cap\alpha)+1$.

Proof. Since $x\in\gamma'\setminus\beta'$, $\beta'\neq\gamma'$. So, $\beta'\cap\gamma'$ is a hyperplane of both β' and γ' and $\dim(\gamma'\cap\alpha) \leq \dim(\beta'\cap\gamma'\cap\alpha)+1 \leq \dim(\beta'\cap\alpha)+1$. We will now prove that $\dim(\beta'\cap\alpha)+1 \leq \dim(\gamma'\cap\alpha)$. If $x\in Q^-(2n+1,\mathbb{K})$ were collinear on $Q^+(2n+1,\mathbb{K}')$ with every point of β', then x would also be collinear on $Q^-(2n+1,\mathbb{K})$ with every point of β, contradicting the fact that $x\in Q^-(2n+1,\mathbb{K})\setminus\beta$ and β is a generator of $Q^-(2n+1,\mathbb{K})$. Hence, the points of β' collinear on $Q^+(2n+1,\mathbb{K}')$ with x form a hyperplane of β' which necessarily coincides with $(\beta\cap\gamma)'$. Since every point of $\beta'\cap\alpha$ is collinear on $Q^+(2n+1,\mathbb{K}')$ with $x\in\alpha$, $\beta'\cap\alpha\subseteq(\beta\cap\gamma)'=\beta'\cap\gamma'$. Hence, $\beta'\cap\alpha\subseteq\gamma'\cap\alpha$. Now, since $x\in(\gamma'\cap\alpha)\setminus(\beta'\cap\alpha)$, we have $\dim(\gamma'\cap\alpha)\geq\dim(\beta'\cap\alpha)+1$. ∎

Suppose x is a point of $\alpha\cap\alpha^\theta\cap\mathrm{PG}(2n+1,\mathbb{K})$, where $n\geq 3$. The subspaces of $Q^-(2n+1,\mathbb{K})$ (respectively $Q^+(2n+1,\mathbb{K}')$) through x define a polar space $Q^-(2n-1,\mathbb{K})$ (respectively $Q^+(2n-1,\mathbb{K}')$). The maximal subspaces of $Q^-(2n+1,\mathbb{K})$ through x form a max $M\cong DQ^-(2n-1,\mathbb{K})$ of $DQ^-(2n+1,\mathbb{K})$. Since α is a maximal subspace of $Q^+(2n-1,\mathbb{K}')$, we can define a valuation f_α^M of M, similarly as we could define the valuation f_α of $DQ^-(2n+1,\mathbb{K})$ from the maximal subspace α of $Q^+(2n+1,\mathbb{K}')$. From Lemma 5.3, we immediately obtain:

Proposition 5.4 *Let f_α^M be as defined before this proposition. Then the valuation f_α of $DQ^-(2n+1,\mathbb{K})$ is the extension of the valuation f_α^M of M.*

Proposition 5.5 *(i) If $n=2$, then f_α is a classical or ovoidal valuation of $DQ^-(5,\mathbb{K})$.*

(ii) If $n=3$, then the valuation f_α of $DQ^-(7,\mathbb{K})$ is either a classical valuation or the extension of an ovoidal valuation of a quad of $DQ^-(7,\mathbb{K})$.

Proof. (i) If $n=2$, then α is a generator of $Q^+(5,\mathbb{K}')$. Since α and α^θ belong to different systems of generators of $Q^+(5,\mathbb{K}')$, $\alpha\cap\alpha^\theta$ is either a line or the empty set. If $\alpha\cap\alpha^\theta$ is a line, then f_α is a classical valuation of $DQ^-(5,\mathbb{K})$ by Lemma 3.1 and Proposition 5.2. Suppose therefore that $\alpha\cap\alpha^\theta=\emptyset$. Then $\dim(\beta'\cap\alpha)\leq 0$ for every generator (= line) β of $Q^-(5,\mathbb{K})$. It follows that f_α can only attain the values 0 and 1. This implies that f_α is an ovoidal valuation of $DQ^-(5,\mathbb{K})$.

(ii) If $n=3$, then since α and α^θ belong to different systems of generators of $Q^+(7,\mathbb{K}')$, $\dim(\alpha\cap\alpha^\theta)\in\{0,2\}$. By Lemma 3.1, there exists a point $x\in\alpha\cap\alpha^\theta\cap\mathrm{PG}(2n+1,\mathbb{K})$. Claim (ii) follows from Claim (i) and Proposition 5.4. (Notice that extensions of classical valuations are again classical.) ∎

Proposition 5.6 *The valuation f_α is the possibly trivial extension of an SDPS-valuation of a convex subspace of $DQ^-(2n+1,\mathbb{K})$.*

Proof. Let $DQ^+(2n+1,\mathbb{K}')$ denote the dual polar space associated to $Q^+(2n+1,\mathbb{K}')$ and let $\mathrm{d}^+(\cdot,\cdot)$ denote the distance function in $DQ^+(2n+1,\mathbb{K}')$.

By Proposition 5.5, the proposition holds if $n\leq 3$. So, suppose $n\geq 4$. Let U denote an arbitrary hex of $DQ^-(2n+1,\mathbb{K})$ corresponding to an $(n-4)$-dimensional subspace β of $Q^-(2n+1,\mathbb{K})$. The subspace β' of $Q^+(2n+1,\mathbb{K}')$

corresponds to a convex subspace F of diameter 4 of $DQ^+(2n+1,\mathbb{K}')$. Let $\widetilde{\alpha}$ denote the unique point of F nearest to α. For every generator γ of $Q^-(2n+1,\mathbb{K})$ through β, put

$$\widetilde{f}_{\widetilde{\alpha}}(\gamma) \;=\; \widetilde{M} - \dim(\gamma' \cap \widetilde{\alpha}),$$

where

$$\widetilde{M} \;:=\; \max\{\dim(\eta' \cap \widetilde{\alpha}) \,|\, \eta \text{ is a generator of } Q^-(2n+1,\mathbb{K}) \text{ through } \beta\}.$$

Then $\widetilde{f}_{\widetilde{\alpha}}$ is a valuation of U, which by Proposition 5.5 is either a classical valuation or the extension of an ovoidal valuation of a quad of U.

Now, for every generator γ of $Q^-(2n+1,\mathbb{K})$ through β, $n-1-\dim(\gamma'\cap\alpha)$ is equal to the distance $\mathrm{d}^+(\gamma',\alpha)$ between the line γ' of $DQ^+(2n+1,\mathbb{K}')$ and the point α of $DQ^+(2n+1,\mathbb{K}')$. Since F is classical in $DQ^+(2n+1,\mathbb{K}')$, $\mathrm{d}^+(\gamma',\alpha) = \mathrm{d}^+(\gamma',\widetilde{\alpha}) + \mathrm{d}^+(\widetilde{\alpha},\alpha)$ and hence $\dim(\gamma'\cap\alpha) = n-1-\mathrm{d}^+(\gamma',\alpha) = n-1-\mathrm{d}^+(\gamma',\widetilde{\alpha})-\mathrm{d}^+(\widetilde{\alpha},\alpha) = \dim(\gamma'\cap\widetilde{\alpha})-\mathrm{d}^+(\widetilde{\alpha},\alpha)$. So, $f_\alpha(\gamma) = M-\dim(\gamma'\cap\alpha) = M+\mathrm{d}^+(\widetilde{\alpha},\alpha)-\dim(\gamma'\cap\widetilde{\alpha}) = M+\mathrm{d}^+(\widetilde{\alpha},\alpha)-\widetilde{M}+\widetilde{f}_{\widetilde{\alpha}}(\gamma)$. It follows that $\widetilde{f}_{\widetilde{\alpha}}$ is the valuation of U induced by f_α. Since U was arbitrary, every induced hex-valuation is either classical or the extension of an ovoidal valuation of a quad. By Proposition 2.4, it now follows that f_α is the possibly trivial extension of an SDPS-valuation of a convex subspace of $DQ^-(2n+1,\mathbb{K})$. ■

Definition. Let F_α denote the convex subspace of $DQ^-(2n+1,\mathbb{K})$ such that f_α is the extension of an SDPS-valuation of F_α. Let X_α denote the SDPS-set of F_α corresponding to the SDPS-valuation of F_α giving rise to f_α. The set X_α consists of those points of $DQ^-(2n+1,\mathbb{K})$ whose f_α-value is equal to 0, or equivalently, consists of those generators γ of $Q^-(2n+1,\mathbb{K})$ for which $\dim(\gamma'\cap\alpha)$ attains its maximal value M.

Proposition 5.7 *F_α is the convex subspace of $DQ^-(2n+1,\mathbb{K})$ corresponding to the subspace $(\alpha\cap\alpha^\theta)\cap\mathrm{PG}(2n+1,\mathbb{K})$ of $Q^-(2n+1,\mathbb{K})$ and X_α is a geometric SDPS-set in F_α.*

Proof. (i) Suppose first that $\alpha\cap\alpha^\theta=\emptyset$. Then n is even by Lemma 3.2. Recall that M is the maximal value of $\dim(\gamma'\cap\alpha)$, where γ ranges over all generators of $Q^-(2n+1,\mathbb{K})$. Let H denote the set of points x of α which are collinear on $Q^+(2n+1,\mathbb{K}')$ with x^θ. Then by Proposition 4.6, H is a nonsingular θ-Hermitian variety of Witt index $\frac{n}{2}$ in α. The set X of generators of $Q^-(2n+1,\mathbb{K})$ of the form $\langle\beta,\beta^\theta\rangle\cap\mathrm{PG}(2n+1,\mathbb{K})$ where β is some generator of H is a (geometric) SDPS-set of $DQ^-(2n+1,\mathbb{K})$.

If γ is a generator of $Q^-(2n+1,\mathbb{K})$, then $\gamma' \cap \alpha$ and $(\gamma' \cap \alpha)^\theta = \gamma' \cap \alpha^\theta$ are disjoint subspaces of γ'. Since $\dim(\gamma') = n-1$, we necessarily have $\dim(\gamma' \cap \alpha) \leq \frac{n}{2} - 1$. Hence, $M \leq \frac{n}{2} - 1$.

If β is a generator of H, then $\gamma = \langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$ is a generator of $Q^-(2n+1,\mathbb{K})$. Moreover, $\gamma' \cap \alpha = \langle \beta, \beta^\theta \rangle \cap \alpha = \beta$ has dimension $\frac{n}{2} - 1$.

So, we can conclude that $M = \frac{n}{2} - 1$. It is clear from the above that the generators γ of $Q^-(2n+1,\mathbb{K})$ for which $\dim(\gamma' \cap \alpha) = \frac{n}{2} - 1$ are precisely those generators of $Q^-(2n+1,\mathbb{K})$ which are of the form $\langle \beta, \beta^\theta \rangle \cap \mathrm{PG}(2n+1,\mathbb{K})$ for some generator β of H. So, $X_\alpha = X$ is a geometric SDPS-set of $DQ^-(2n+1,\mathbb{K})$. Since $DQ^-(2n+1,\mathbb{K})$ is the convex subspace of $DQ^-(2n+1,\mathbb{K})$ corresponding to the subspace $\emptyset = \alpha \cap \alpha^\theta \cap \mathrm{PG}(2n+1,\mathbb{K})$ of $Q^-(2n+1,\mathbb{K})$, we have proved our claim.

(ii) Suppose $\beta := \alpha \cap \alpha^\theta \cap \mathrm{PG}(2n+1,\mathbb{K}) \neq \emptyset$. Let F denote the convex subspace of $DQ^-(2n+1,\mathbb{K})$ corresponding to β. By successive application of Proposition 5.4, we see that f_α is the extension of a valuation f_α^F of F. So, X_α must be a set of points of F. Now, taking the quotient polar spaces P and P' obtained by considering all subspaces of $Q^-(2n+1,\mathbb{K})$ and $Q^+(2n+1,\mathbb{K}')$ through β and β', respectively, and applying (i), we see that X_α must be a geometric SDPS-set in F. ■

Now, put

$$K \ := \ \frac{n-1-\dim(\alpha \cap \alpha^\theta)}{2}.$$

Then $K \in \mathbb{N}$ by Lemma 3.2. More precisely, we have $0 \leq K \leq \lfloor \frac{n}{2} \rfloor$. By Proposition 5.7, the diameter $\mathrm{diam}(F_\alpha)$ of F_α is equal to $(n-1) - \dim(\alpha \cap \alpha^\theta) = 2K$. So, the maximal value of an SDPS-valuation of F_α is equal to K. It follows that the maximal value of f_α is equal to

$$K + \mathrm{diam}(DQ^-(2n+1,\mathbb{K})) - \mathrm{diam}(F_\alpha) = n - K.$$

In the following proposition, we determine the precise value of the parameter $M = \max\{\dim(\eta' \cap \alpha) \,|\, \eta \text{ is a generator of } Q^-(2n+1,\mathbb{K})\}$.

Proposition 5.8 *We have $M = n - K - 1$.*

Proof. We have $\dim(\alpha \cap \alpha^\theta) = n - (2K+1)$. Let π be the $(n-2K-1)$-dimensional subspace of $\mathrm{PG}(2n+1,\mathbb{K})$ such that $\pi' = \alpha \cap \alpha^\theta$. Taking the quotient of $Q^-(2n+1,\mathbb{K})$ and $Q^+(2n+1,\mathbb{K}')$ over the respective subspaces π and $\pi' = \alpha \cap \alpha^\theta$, we obtain polar spaces isomorphic to $Q^-(4K+1,\mathbb{K})$ and $Q^+(4K+1,\mathbb{K}')$. By successive application of Lemma 5.3, we see that $M = \dim(\pi') + 1 +$ dimension of a generator of $H(2K,\mathbb{K}',\theta) = n-K-1$. (Recall that by the proof of Proposition 5.7, M equals the dimension of a generator of H if $\alpha \cap \alpha^\theta = \emptyset$.) ∎

Corollary 5.9 *There exists a generator η of $Q^-(2n+1,\mathbb{K})$ such that $\eta' \cap \alpha = \emptyset$.*

Proof. For every generator η of $Q^-(2n+1,\mathbb{K})$, we have $f_\alpha(\eta) = M - \dim(\eta' \cap \alpha) = n - K - 1 - \dim(\eta' \cap \alpha)$. The claim now follows from the fact that the maximal value of $f_\alpha(\eta)$ is equal to $n - K$. ∎

Now, let $\epsilon \in \{+,-\}$ such that $\alpha \in \mathcal{M}^\epsilon$. Recall that $HS^\epsilon(2n+1,\mathbb{K}')$ denotes the half-spin geometry for $Q^+(2n+1,\mathbb{K}')$ defined on the set $\mathcal{M}^\epsilon$. Let $\mathrm{d}_\epsilon(\cdot,\cdot)$ denote the distance function in $HS^\epsilon(2n+1,\mathbb{K}')$. For any two elements $\alpha_1, \alpha_2 \in \mathcal{M}^\epsilon$, we have $\mathrm{d}_\epsilon(\alpha_1,\alpha_2) = \frac{n-\dim(\alpha_1\cap\alpha_2)}{2}$. The diameter of $HS^\epsilon(2n+1,\mathbb{K}')$ is equal to $\lfloor \frac{n+1}{2} \rfloor$.

For every generator γ of $Q^-(2n+1,\mathbb{K})$, let γ^ϕ denote the unique element of $\mathcal{M}^\epsilon$ through γ'. Then ϕ defines a full embedding of $\Delta = DQ^-(2n+1,\mathbb{K})$ into $HS^\epsilon(2n+1,\mathbb{K}')$ (the natural embedding of $DQ^-(2n+1,\mathbb{K})$ into $HS^\epsilon(2n+1,\mathbb{K}')$). Since γ^ϕ and α belong to the same system of generators of $Q^+(2n+1,\mathbb{K}')$, $n-\dim(\alpha\cap\gamma^\phi)$ is even. Obviously, $\dim(\alpha\cap\gamma^\phi)-\dim(\alpha\cap\gamma') \in \{0,1\}$. Hence, $n - \dim(\alpha\cap\gamma^\phi) = 2\cdot\lfloor\frac{n-\dim(\alpha\cap\gamma')}{2}\rfloor$, i.e.

$$\mathrm{d}_\epsilon(\alpha,\gamma^\phi) \;=\; \lfloor \frac{n-\dim(\alpha\cap\gamma')}{2} \rfloor.$$

Since the maximal value of $\dim(\alpha\cap\eta')$, where η ranges over all generators of $Q^-(2n+1,\mathbb{K})$, is equal to $n-K-1$, we have

$$\mathrm{d}_\epsilon(\alpha,\Delta^\phi) \;=\; \lfloor \frac{K+1}{2} \rfloor.$$

Now, for every generator γ of $Q^-(2n+1,\mathbb{K})$,

$$\begin{aligned}
\mathrm{d}_\epsilon(\alpha,\gamma^\phi) &= \lfloor \frac{n-\dim(\alpha\cap\gamma')}{2} \rfloor \\
&= \lfloor \frac{n-M+M-\dim(\alpha\cap\gamma')}{2} \rfloor \\
&= \lfloor \frac{K+1+f_\alpha(\gamma)}{2} \rfloor \\
&= \lfloor \frac{K+1+\mathrm{d}_\Delta(\gamma,X_\alpha)}{2} \rfloor.
\end{aligned}$$

This proves Theorem 1.3.

Remark. Since the maximal value of $f_\alpha(\gamma) = \mathrm{d}_\Delta(\gamma, X_\alpha)$ is equal to $n - K$, the maximal value of $\mathrm{d}_\epsilon(\alpha, \gamma^\phi)$ is equal to $\lfloor \frac{n+1}{2} \rfloor$ which is precisely the diameter of $HS^\epsilon(2n+1, \mathbb{K}')$.

We will now prove Theorem 1.4. We will need the following lemma, which is easy to prove, see e.g. [9, Lemma 2.5].

Lemma 5.10 *(i) If n is odd, then the set of elements of $\mathcal{M}^\epsilon$ meeting a given element of $\mathcal{M}^\epsilon$ is a hyperplane of $HS^\epsilon(2n+1, \mathbb{K}')$.*

(ii) If n is even, then the set of elements of $\mathcal{M}^\epsilon$ meeting a given element of $\mathcal{M}^{-\epsilon}$ is a hyperplane of $HS^\epsilon(2n+1, \mathbb{K}')$.

Now, suppose that n is even, that $\alpha \in \mathcal{M}^\epsilon$ and that $\alpha \cap \alpha^\theta = \emptyset$. Then by Lemma 5.10 (ii) and Lemma 3.2, the set of elements of $\mathcal{M}^\epsilon$ meeting α^θ defines a hyperplane H_α of $HS^\epsilon(2n+1, \mathbb{K}')$.

Let γ be an arbitrary generator of $Q^-(2n+1, \mathbb{K})$. Then $\gamma^\phi \in H_\alpha$ if and only if $\dim(\alpha^\theta \cap \gamma^\phi) \geq 0$. Now, $\dim(\alpha^\theta \cap \gamma^\phi) - \dim(\alpha^\theta \cap \gamma') \in \{0, 1\}$ and $\dim(\alpha^\theta \cap \gamma^\phi)$ is odd since α^θ and γ^ϕ belong to different systems of generators. It follows that $\gamma^\phi \in H_\alpha$ if and only if $\dim(\alpha^\theta \cap \gamma') \geq 0$. Now,

$$\begin{aligned} \dim(\alpha^\theta \cap \gamma') &= \dim(\alpha \cap \gamma') \\ &= M - f_\alpha(\gamma) \\ &= n - K - 1 - f_\alpha(\gamma). \end{aligned}$$

So, $\gamma^\phi \in H_\alpha$ if and only if $f_\alpha(\gamma) \leq n - K - 1$. Now, the maximal value of the valuation f_α is equal to $n - K$. So, $\gamma^\phi \in H_\alpha$ if and only if γ belongs to the hyperplane of $DQ^-(2n+1, \mathbb{K})$ associated to the SDPS-set X_α. Now, let e denote the spin embedding of $HS^\epsilon(2n+1, \mathbb{K}')$. Then by the main result of Shult [16] (see also Corollary 1.3 of [8] for an alternative proof) every hyperplane of $HS^\epsilon(2n+1, \mathbb{K}')$ arises from e. In particular, H_α arises from e. Now, the map $e \circ \phi$ defines a full embedding e' of $DQ^-(2n+1, \mathbb{K})$ which is isomorphic to the spin embedding of $DQ^-(2n+1, \mathbb{K})$. Since H_α arises from e, the hyperplane of $DQ^-(2n+1, \mathbb{K})$ associated to the SDPS-set X_α arises from e'. This proves Theorem 1.4.

6 Appendix: An alternative construction for the unique geometric SDPS-set of $DQ^-(2n+1, \mathbb{K})$

In De Bruyn and Vandecasteele [11] a construction was given to obtain SDPS-sets of the dual polar space $DQ^-(2n+1, q)$. We recall this construction.

Consider the finite field $\mathbb{F}_{q^2}$ with q^2 elements and let $\mathbb{F}_q$ denote the subfield of order q of $\mathbb{F}_{q^2}$. Let δ denote an arbitrary element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Consider the following bijection ϕ between the vector spaces $\mathbb{F}_q^{4n+2}$ and $\mathbb{F}_{q^2}^{2n+1}$:

$$\phi(X_1, X_2, \ldots, X_{4n+2}) = (X_1 + \delta X_2, \ldots, X_{4n+1} + \delta X_{4n+2}).$$

Let $\langle \cdot, \cdot \rangle$ be a nondegenerate Hermitian form of $\mathbb{F}_{q^2}^{2n+1}$. For every $\bar{x} \in \mathbb{F}_{q^2}^{2n+1}$, we define $h(\bar{x}) := \langle \bar{x}, \bar{x} \rangle$ and for every $\bar{x} \in \mathbb{F}_q^{4n+2}$, we define $q(\bar{x}) := \langle \phi(\bar{x}), \phi(\bar{x}) \rangle$. The equation $h(\bar{x}) = 0$, respectively $q(\bar{x}) = 0$, defines a nonsingular Hermitian variety $H(2n, q^2)$ in $\mathrm{PG}(2n, q^2)$, respectively a nonsingular elliptic quadric $Q^-(4n+1, q)$ in $\mathrm{PG}(4n+1, q)$. With every generator of $H(2n, q^2)$, there corresponds (via the map ϕ^{-1}) a generator of $Q^-(4n+1, q)$. The set of generators of $Q^-(4n+1, q)$ which arise in this way is an SDPS-set of $DQ^-(4n+1, q)$.

We will now show that the SDPS-sets which arise in this way are geometric. In order to facilitate the proof, we will give a slightly different but equivalent construction (which is presented here for possibly infinite fields).

Let $\mathbb{K}$, $\mathbb{K}'$, θ, n, $V(2n+2, \mathbb{K})$, $V(2n+2, \mathbb{K}')$, $\mathrm{PG}(2n+1, \mathbb{K})$ and $\mathrm{PG}(2n+1, \mathbb{K}')$ be as in Section 3 and suppose that n is even. Let V be an $(n+1)$-dimensional subspace of $V(2n+2, \mathbb{K}')$ such that $V^\theta \cap V = \{\bar{o}\}$. Let $\{\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n\}$ be a basis of V and let α be the subspace of $\mathrm{PG}(2n+1, \mathbb{K}')$ corresponding to V. For every $i \in \{0, \ldots, n\}$, we define

$$\begin{aligned} \bar{f}_i &:= \bar{e}_i + \bar{e}_i^\theta, \\ \bar{g}_i &:= \delta \cdot \bar{e}_i + \delta^\theta \cdot \bar{e}_i^\theta, \end{aligned}$$

where δ is some given element of $\mathbb{K}' \setminus \mathbb{K}$. Then $\{\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_n, \bar{g}_0, \bar{g}_1, \ldots, \bar{g}_n\}$ is a basis of $V(2n+2, \mathbb{K})$. Define the following bijection ϕ between $V(2n+2, \mathbb{K})$ and V:

$$\phi\Big(\sum_{i=0}^{n} (X_i \bar{f}_i + Y_i \bar{g}_i)\Big) \ := \ \sum_{i=0}^{n} (X_i + \delta Y_i) \bar{e}_i.$$

The following claim is obvious:

Claim I: *For every vector $\bar{x} \neq \bar{o}$ of $V(2n+2, \mathbb{K})$, the point $\langle \bar{x} \rangle$ of $\mathrm{PG}(2n+1, \mathbb{K}')$ is contained on the line connecting the points $\langle \phi(\bar{x}) \rangle \in \alpha$ and $\langle \phi(\bar{x})^\theta \rangle \in \alpha^\theta$.*

Now, let $< \cdot, \cdot >$ be a nondegenerate θ-Hermitian form of V of maximal Witt index $\frac{n}{2}$. For every vector $\bar{x}$ of $V(2n+2, \mathbb{K})$, we define

$$q(\bar{x}) \ := \ \langle \phi(\bar{x}), \phi(\bar{x}) \rangle.$$

Claim II: *q is a nondegenerate quadratic form of Witt index n of $V(2n+2,\mathbb{K})$.*

PROOF. For every $\bar{x} \in V(2n+2,\mathbb{K})$ and every $k \in \mathbb{K}$, we have $q(k\bar{x}) = k^2 q(\bar{x})$. Now, for all $\bar{x}_1, \bar{x}_2 \in V(2n+2,\mathbb{K})$, we put $B(\bar{x}_1,\bar{x}_2) := q(\bar{x}_1+\bar{x}_2) - q(\bar{x}_1) - q(\bar{x}_2) = \langle \phi(\bar{x}_1), \phi(\bar{x}_2)\rangle + \langle \phi(\bar{x}_2), \phi(\bar{x}_1)\rangle$. Obviously, B is a symmetric $\mathbb{K}$-bilinear form on $V(2n+2,\mathbb{K})$. We prove that B is nondegenerate. If B were degenerate, then there exists an $\bar{x}^* \in V(2n+2,\mathbb{K}) \setminus \{\bar{0}\}$ such that $\kappa(\bar{y}) := \langle \phi(\bar{x}^*), \bar{y}\rangle + \langle \bar{y}, \phi(\bar{x}^*)\rangle = 0$ for all $\bar{y} \in V$. From $\kappa(\bar{y}) = \kappa(\delta\bar{y}) = 0$, it then follows that $\langle \phi(\bar{x}^*), \bar{y}\rangle = 0$ for all $\bar{y} \in V$. This contradicts the fact that $\langle \cdot, \cdot \rangle$ is nondegenerate.

If U is a subspace of vector dimension $\frac{n}{2}$ of V which is totally isotropic with respect to $\langle \cdot,\cdot\rangle$, then $q(\bar{x}) = 0$ for every $\bar{x} \in \phi^{-1}(U)$. Hence, the Witt index of q is at least n. Suppose the Witt index of q is bigger than n. Then there exists a vector $\bar{x}^* \in V(2n+2,\mathbb{K})$ not belonging to $\phi^{-1}(U)$ such that $q(\bar{x}) = 0$ for every vector $\bar{x}$ belonging to the subspace of $V(2n+2,\mathbb{K})$ generated by $\bar{x}^*$ and $\phi^{-1}(U)$. This implies that $\kappa(\bar{y}) := \langle \phi(\bar{x}^*), \bar{y}\rangle + \langle \bar{y}, \phi(\bar{x}^*)\rangle$ for any $\bar{y} \in U$. From $\kappa(\bar{y}) = \kappa(\delta\bar{y}) = 0$, it then follows that $\langle \phi(\bar{x}^*), \bar{y}\rangle = 0$ for any $\bar{y} \in U$. Since also $\langle \phi(\bar{x}^*), \phi(\bar{x}^*)\rangle = q(\bar{x}^*) = 0$ and $\phi(\bar{x}^*) \notin U$, this contradicts the fact that U is a maximal totally isotropic subspace of V.

So, q is a nondegenerate quadratic form of Witt index n in $V(2n+2,\mathbb{K})$.

■

So, with q there is associated a nonsingular quadric Q of Witt index n in $\mathrm{PG}(2n+1,\mathbb{K})$ and a quadric $\widetilde{Q}$ in $\mathrm{PG}(2n+1,\mathbb{K}')$. Since the bilinear form associated to q is nondegenerate, $\widetilde{Q}$ is a nonsingular quadric of Witt index $n' \in \{n, n+1\}$ in $\mathrm{PG}(2n+1,\mathbb{K}')$. Let $\widetilde{H}$ be the θ-Hermitian variety of α associated to $\langle\cdot,\cdot\rangle$ and let ζ denote the Hermitian polarity of α associated to $\langle\cdot,\cdot\rangle$. We will prove that $\alpha \subseteq \widetilde{Q}$. Let $p = \langle \bar{y}\rangle$ be an arbitrary point of α.

(a) Suppose first that $p \in \widetilde{H}$. By Claim I, for every point $\langle \bar{x}\rangle \in pp^\theta \cap \mathrm{PG}(2n+1,\mathbb{K})$, $\phi(\bar{x})$ is a multiple of $\bar{y}$. Hence, the line $pp^\theta \cap \mathrm{PG}(2n+1,\mathbb{K})$ of $\mathrm{PG}(2n+1,\mathbb{K})$ is completely contained in $Q^-(2n+1,\mathbb{K})$. So, $pp^\theta \subseteq \widetilde{Q}$. In particular, $p \in \widetilde{Q}$.

(b) Suppose $p \in \alpha \setminus \widetilde{H}$. Clearly, there exists a point $r \in \widetilde{H} \setminus p^\zeta$. For such a point r, $rp \cap \widetilde{H}$ is a Baer subline of rp. Since each of the $|\mathbb{K}|+1 \geq 3$ points of $rp \cap \widetilde{H}$ are contained in $\widetilde{Q}$, the whole line rp is contained in $\widetilde{Q}$. In particular, $p \in \widetilde{Q}$.

Since $\widetilde{Q}$ contains subspaces of projective dimension n, the Witt index of $\widetilde{Q}$ must be equal to $n+1$. In the sequel, we will denote Q by $Q^-(2n+1,\mathbb{K})$ and $\widetilde{Q}$ by $Q^+(2n+1,\mathbb{K}')$. Now, let H denote the set of all points p of α which are collinear on $Q^+(2n+1,\mathbb{K}')$ with p^θ. Clearly, $p \in H$ if and only if every point of $pp^\theta \cap \mathrm{PG}(2n+1, \mathbb{K})$ belongs to $Q^-(2n+1, \mathbb{K})$. Now, a point $\langle \bar{x} \rangle \in pp^\theta \cap \mathrm{PG}(2n+1, \mathbb{K})$ belongs to $Q^-(2n+1, \mathbb{K})$ if and only if $p = \langle \phi(\bar{x}) \rangle \in \widetilde{H}$. It follows that $H = \widetilde{H}$. Hence, the geometric SDPS-set of $DQ^-(2n+1, \mathbb{K})$ associated to α coincides with the set of all generators $\langle U \rangle$ of $Q^-(2n+1, \mathbb{K})$ for which $\phi(U)$ is a maximal totally isotropic subspace of V with respect to the Hermitian form $\langle \cdot, \cdot \rangle$. This is precisely what we needed to prove.

Remark. Although both constructions give rise to isomorphic SDPS-sets, there is an important difference between them. The construction described in this paper allows to obtain many (geometric) SDPS-sets in a *given* dual polar space isomorphic to $DQ^-(2n+1, \mathbb{K})$. The other construction allows to obtain an SDPS-set in *some* dual polar space isomorphic to $DQ^-(2n+1, \mathbb{K})$.

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